

c-Theorem for Disordered Systems

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We find an analog of Zamolodchikov's *c*-theorem for disordered two dimensional noninteracting systems in their supersymmetric representation. For this purpose we introduce a new parameter b which flows along the renormalization group trajectories much like the central charge for unitary two dimensional field theories. However, it is not known yet if this flow is irreversible. b turns out to be related to the central extension of a certain algebra, a generalization of the Virasoro algebra, which we show may be present at the critical points of these theories. b is also related to the physical free energy of the disordered system defined on a cylinder. We discuss possible applications by computing b for two dimensional Dirac fermions with random gauge potential.

A basic fact in the theory of disordered noninteracting systems is that they can be described by the special supersymmetric field theories, the theories which contain equal number of bosonic and fermionic fields and which are invariant under *supergroups*, the groups which rotate bosonic and fermionic fields into each other [1,2]. Unfortunately it is generally very hard to solve these theories. Perturbative methods remain the only tool used consistently on them. One of the main difficulties in treating these theories stems from the fact that they are nonunitary, having been constructed in such a way as to make their partition function equal to 1.

Nevertheless there is certain hope that in two dimensions these theories must be treatable. After all, they are not much different from other two dimensional field theories whose critical points are described by conformal field theory [3]. It goes without saying that solving these theories is very important for comparison with experiment [1].

It can be shown (see below) that if a conformal field theory describes a critical point of one of these supersymmetric theories, its central charge c must be equal to 0. There is a large number of conformal field theories whose central charge is equal to 0, the direct product of two conformal field theories with central charge equal to an arbitrary number c and to $-c$ being just one example. Therefore, the central charge loses its meaning as a way to distinguish between different critical points of these theories.

In this paper we introduce a parameter b which replaces the central charge as a way to distinguish between different critical points of these supersymmetric theories. c is equal to zero because it counts the difference between bosonic and fermionic degrees of freedom, which is 0 for those theories. Roughly speaking b counts the *sum* of bosonic and fermionic degrees of freedom. Moreover, we show that b flows along the renormalization group trajectories in a way similar to c in unitary theories, as expressed in the so-called *c*-theorem [4].

A generic supersymmetric disordered field theory is invariant under the group which mixes bosons and fermions. We will choose this group to be $U(1|1)$, even though the actual symmetry group of our theory could be larger or could be based on other groups [1,2]. We will denote the four generators of this group by J , j , η and $\bar{\eta}$, following [5]. As discussed in appendix A the action of such a field theory is a part of the supermultiplet consisting of S , s , ζ , and $\bar{\zeta}$, the first two being commuting and the second two anticommuting quantities. As a part of the supermultiplet the action itself is a variation of either ζ or $\bar{\zeta}$ under the action of either $\bar{\eta}$ or η respectively. Correspondingly the energy-momentum tensor $T_{\mu\nu}$ is also a part of the supermultiplet consisting of $T_{\mu\nu}$, $t_{\mu\nu}$, $\xi_{\mu\nu}$ and $\bar{\xi}_{\mu\nu}$. Because of this, the energy-momentum tensor is a variation of either ξ or $\bar{\xi}$ under the action of either $\bar{\eta}$ or η respectively.

The theory we are considering can be thought of as a topological field theory. The topological field theories are defined as theories with expectation values of the products of any number of the energy momentum tensors equal to 0, and indeed for our theory

$$\langle T_{\mu\nu}(x)T_{\mu'\nu'}(x')\dots \rangle = \delta_{\bar{\eta}} \langle \xi_{\mu\nu}(x)T_{\mu'\nu'}(x')\dots \rangle = 0 \quad (1)$$

which is consistent with the fact that the partition function of this theory is equal to 1.

Now it follows from (1) that if such a field theory has a critical point, its central charge will be equal to zero because the central charge is related to the expectation value of the products of energy momentum tensors [3]. In fact, Zamolodchikov's *c*-function [4], which is equal to the central charge at the critical point and usually changes along the renormalization group trajectories, will be equal to 0 everywhere. Therefore, the central charge is not a useful way to parametrize such a field theory.

However, the correlation function of the energy momentum tensor $T_{\mu\nu}$ with $t_{\mu\nu}$ does not have to be zero. At this point, following Zamolodchikov [4] we introduce the following notations

$$\langle T(z, \bar{z})t(0, 0) \rangle = \frac{F(z\bar{z})}{z^4}, \quad \langle T(z, \bar{z})\theta(0, 0) \rangle = \frac{G(z\bar{z})}{z^3\bar{z}}, \quad \langle \Theta(z, \bar{z})\theta(0, 0) \rangle = \frac{H(z\bar{z})}{z^2\bar{z}^2} \quad (2)$$

where $T \equiv T_{zz}$, $\Theta \equiv T_{z\bar{z}}$ and $t \equiv t_{zz}$, $\theta \equiv t_{z\bar{z}}$.

The correlation functions (2) (and their complex conjugate counterparts) exhaust all the possible correlations between $T_{\mu\nu}$ and $t_{\mu\nu}$ by virtue of the following remarkable identity

$$\langle T_{\mu\nu}(x)t_{\mu'\nu'}(0) \rangle = \langle t_{\mu\nu}(x)T_{\mu'\nu'}(0) \rangle. \quad (3)$$

Its proof is discussed in the appendix B. It allows to relate various correlation functions, for example, $\langle T(z, \bar{z})\theta(0) \rangle = \langle \Theta(z, \bar{z})t(0) \rangle$.

Now the tensors $\xi_{\mu\nu}$, $\bar{\xi}_{\mu\nu}$ and $t_{\mu\nu}$ are not necessarily conserved. However, combining the fact that $T_{\mu\nu}$ is conserved with the identity (3) we can show that

$$\langle \partial_\mu t_{\mu\nu}(x)T_{\mu'\nu'}(0) \rangle = \langle \partial_\mu T_{\mu\nu}(x)t_{\mu'\nu'}(0) \rangle = 0. \quad (4)$$

Therefore, as far as the correlators (2) are concerned, we can work with t as if it were a conserved tensor. This allows us to derive the analog of Zamolodchikov's c -theorem.

We define

$$b = F - 2G - 3H. \quad (5)$$

Now we are in the position to claim that the following relation is true,

$$\frac{db(t)}{dt} = -6H \quad (6)$$

where t is a parameter which increases along the renormalization group trajectory along the infrared direction. Indeed, Zamolodchikov considered a function $c = 2F' - 4G' - 6H'$ where F' , G' , and H' are the correlators different from (2) only by replacing t by T everywhere. He showed that $dc/dt = -12H'$. In his proof the only relevant information was that $T_{\mu\nu}$ was conserved. Fortunately by virtue of (4) we can work with $t_{\mu\nu}$ as if it were conserved and arrive at (6) following Zamolodchikov's work step by step.

Does the parameter b decrease along the renormalization group trajectories? We cannot give a definite answer to this question at this time, but we find the following heuristic arguments useful.

b does decrease along the renormalization group trajectories as long as $H > 0$. Can we show that $H > 0$? As shown in appendix C, if the strength of the disorder is equal to 0, the bosonic and fermionic sectors decouple and then the quantity H will coincide with the purely bosonic correlator

$$2 \langle \Theta^b(x)\Theta^b(0) \rangle = \frac{H(z\bar{z})}{z^2\bar{z}^2}, \quad (7)$$

where Θ^b refers to the bosonic subsector of the theory. Since the bosonic subsector is unitary, $H \geq 0$. As we increase the strength of the disorder H may change and in principle can change its sign. Apparently it maintains its sign for small disorder, unless it is a discontinuous function of disorder strength. We also note that H may change sign at the points along the renormalization group trajectory where $\langle \Theta(x)\theta(0) \rangle = 0$.

At the critical point of the theory $H = 0$ because the trace of the energy momentum tensor $\Theta = 0$. Then the parameter b reaches a fixed value, which we will also call b . What is the meaning of b at the critical point? Obviously it coincides with the correlator

$$\langle T(z)t(0) \rangle = \frac{b}{z^4}. \quad (8)$$

By analogy with appendix C, consider a critical disordered theory with the strength of the disorder equal to zero. For example, we can take the theory (C2) and set $E = 0$ to make it critical. Just as in appendix C, $T = T^b + T^f$ and $t = T^b - T^f$ where T^b is the energy momentum tensor of the bosonic sector and T^f is that of the fermionic sector. Then it is obvious that $b = c$ where c is the central charge of the bosonic sector.

Thus b coincides with the central charge of the bosonic theory in the absence of disorder, and it becomes a genuine new parameter as we turn on disorder. Figuratively speaking we can say that while the central charge of the

supersymmetric theory counts the number of bosonic degrees of freedom minus fermionic degrees of freedom, which is always 0, b on the other hand counts the number of bosonic plus the number of fermionic degrees of freedom (which of course is just twice the number of bosonic degrees of freedom).

What does Eq. (8) imply for the conformal theory of the critical point? Since $t_{\mu\nu}$ is not necessarily conserved tensor, t does not have to be holomorphic. However, assuming that it is, we are led to propose the existence of the following algebra for the expansion modes of $T = \sum_n L_n z^{-n-2}$ and $t = \sum_n l_n z^{-n-2}$

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} \\ [L_n, l_m] &= (n-m)l_{n+m} + \frac{b}{6}(n^3 - n)\delta_{n,-m} \\ [l_n, l_m] &= (n-m)L_{n+m} \end{aligned} \tag{9}$$

which generalizes the Virasoro algebra for the critical points of two dimensional disordered systems. The absence of the central extension in the last commutator in (9) stems from the fact that t is defined up to addition of T .

The algebra (9) is certainly present for the Kac-Moody algebras [6] based on the supergroups [5] because t is going to be quadratic in currents and therefore conserved. We compute the parameter b for them in the appendix D.

Does the algebra (9) have to be present for other more generic disordered critical points? The answer to this question depends crucially on whether t is holomorphic at those points. While we lack a general proof that t has to be holomorphic, we can show that in order for the primary operators of this theory to have nonzero correlation functions the holomorphic operator t has to appear on the right hand side of various operator product expansions.

Indeed, consider a primary operator V . The general rules of conformal field theory [3] dictate the operator product expansion of V with itself,

$$V(z)V(0) = \frac{C}{z^{2h_V}} \left(\frac{c}{h_V} + 2z^2 T(0) + \dots \right) + \dots \tag{10}$$

where C is an overall normalization constant, h_V is the dimension of V and c is the central charge (see also [7]). It follows from (10) that the correlation function $\langle V(z)V(0) \rangle$ is proportional to the central charge of the theory,

$$\langle V(z)V(0) \rangle = C \frac{c/h_V}{z^{2h_V}}. \tag{11}$$

Normally the normalization constant C is chosen to be inversely proportional to the central charge to cancel that dependence. However, for the theories considered in this paper $c = 0$ and it looks like all the correlation function of these theories have to be zero! One way this paradox can be resolved is by including a holomorphic field t in the operator product expansion of V with itself,

$$V(z)V(0) = \frac{C}{z^{2h_V}} \left(\frac{2b}{h_V} + 2z^2 t(0) + \dots \right) + \dots \tag{12}$$

because while $L_2 T = 0$, $L_2 t = b$ as follows from (9). This is exactly what happens with the current algebra considered in the appendix D.

The situation may become more complicated if we are led to include zero dimensional logarithmic operators [8] which have unusual expectation values. We are not going to consider this possibility here.

Now what is the physical meaning of b ? According to (9) b is related to the transformation properties of t under conformal transformations. This will allow us to show that b relates the physical free energy of the disordered system on a cylinder to its circumference L as in the following,

$$F_{\text{phys}} = -\frac{\pi b}{6L} \tag{13}$$

Indeed, it is well known [9] that the central charge c of a conformal field theory relates the free energy of the theory on the cylinder to the circumference of the cylinder L ,

$$F = -\frac{\pi c}{6L} \tag{14}$$

The central charge of our theory is 0, which is only natural because the free energy of this theory is also zero, being derived from the partition function which is equal to 1. Nevertheless, there is a well defined physical free energy of the

system, which can be computed as a disorder averaged logarithm of the physical partition function defined without using supersymmetry or replica tricks. It is this free energy that enters (13).

To show that (13) holds, we use that the expectation value of the tensor $t_{\mu\nu}$ of the supersymmetric theory is related to the expectation value of the averaged over disorder physical energy momentum tensor,

$$\langle t_{\mu\nu} \rangle = 2 \langle T_{\mu\nu}^{\text{phys}} \rangle. \quad (15)$$

This relation is discussed in appendix E where a careful definition of the physical free energy is given. Note that a naive generalization of this relation to higher order correlators would not be correct.

On the other hand, the expectation value of t computed on the cylinder with circumference L is given, by standard arguments,

$$\langle t_{\text{cyl}} \rangle = -\frac{b\pi^2}{3L^2} \quad (16)$$

because it follows from (9) that the transformation law for t coincides with the transformation law of the energy momentum tensor in the conformal field theory with $c = 2b$.

From now on, we can easily derive (13) by essentially following Affleck's derivation of (14) in [9] step by step and combining it with (15).

Therefore we see that the parameter b is a way for the supersymmetric system to remember its nonsupersymmetric disordered 'past'.

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APPENDIX A:

Consider the supersymmetric action S describing the disordered system before the average over disorder has been carried out.

$$S = \int \phi^*(E - H)\phi + \psi^*(E - H)\psi \quad (A1)$$

where H is a hamiltonian of a disordered system, including disorder, E is a parameter (energy), ϕ are commuting and ψ are anticommuting variables. A basic property of this action is its invariance under the supergroup $U(1|1)$. This group has four generators, J , j , η , and $\bar{\eta}$, which act on the basic fields ψ and ϕ in the following way [5]

$$\begin{aligned} J : \delta\phi &= \lambda\phi, \delta\phi^* = -\lambda\phi^*, \delta\psi = \lambda\psi, \delta\psi^* = -\lambda\psi^* \\ j : \delta\phi &= \lambda\phi, \delta\phi^* = -\lambda\phi^*, \delta\psi = -\lambda\psi, \delta\psi^* = \lambda\psi^* \\ \eta : \delta\phi &= \epsilon\psi, \delta\psi^* = -\epsilon\phi^* \\ \bar{\eta} : \delta\phi^* &= \epsilon\psi^*, \delta\psi = \epsilon\phi \end{aligned} \quad (A2)$$

where λ is a commuting and ϵ an anticommuting parameter.

These generators obey the following commutation relations

$$\begin{aligned} [j, \eta] &= 2\eta, [j, \bar{\eta}] = -2\bar{\eta} \\ \{\eta, \bar{\eta}\} &= J \end{aligned} \quad (A3)$$

with $[\]$ being the commutator and $\{\ }$ being the anticommutator. All other commutators in this algebra are trivial.

A remarkable property of the action S is that it can be expressed as a variation of some other quantity under either η or $\bar{\eta}$,

$$S = -\delta_\eta \left(\int \psi^*(E - H)\phi \right) = \delta_{\bar{\eta}} \left(\int \phi^*(E - H)\psi \right) \quad (A4)$$

This property remains to be true even after the average with respect to disorder is taken [2]. Moreover, the action S is just a part of a supermultiplet. The other members of the supermultiplet are

$$\begin{aligned}
\zeta &= \int \phi^*(E - H)\psi \\
\bar{\zeta} &= - \int \psi^*(E - H)\phi \\
s &= \int \phi^*(E - H)\phi - \psi^*(E - H)\psi
\end{aligned} \tag{A5}$$

which transform under $U(1|1)$ as η , $\bar{\eta}$, and j . The action itself transforms as J , that is, it does not transform under $U(1|1)$, as in (A3). We also note that after averaging over disorder S , ζ , $\bar{\zeta}$, and s become more complicated nonquadratic functions of the basic fields ϕ and ψ . Nevertheless they preserve their transformation properties.

Analogously if $T_{\mu\nu}$ is an energy momentum tensor of the supersymmetric theory of this kind, it is also part of the supermultiplet and transforms as J . The remaining tensors $\xi_{\mu\nu}$, $\bar{\xi}_{\mu\nu}$ and $t_{\mu\nu}$ transform as η , $\bar{\eta}$ and j .

APPENDIX B:

Let us prove that

$$\langle T_{\mu\nu}(x)t_{\mu'\nu'}(0) \rangle = \langle t_{\mu\nu}(x)T_{\mu'\nu'}(0) \rangle \tag{B1}$$

Consider the following correlation function

$$\langle t_{\mu\nu}(x)\xi_{\mu'\nu'}(0) \rangle \tag{B2}$$

By computing its variation with respect to $\bar{\eta}$ we obtain

$$-2 \langle \bar{\xi}_{\mu\nu}(x)\xi_{\mu'\nu'}(0) \rangle + \langle t_{\mu\nu}(x)T_{\mu'\nu'}(0) \rangle = 0 \tag{B3}$$

Analogously, by computing the variation of $\langle \bar{\xi}_{\mu\nu}(x)t_{\mu'\nu'}(0) \rangle$ under η we get

$$\langle T_{\mu\nu}(x)t_{\mu'\nu'}(0) \rangle - 2 \langle \bar{\xi}_{\mu\nu}(x)\xi_{\mu'\nu'}(0) \rangle = 0 \tag{B4}$$

Combining (B3) and (B4) we arrive at (B1).

APPENDIX C:

Consider a field theory corresponding to the disordered hamiltonian

$$H = -\Delta + V(x) \tag{C1}$$

where we set the disorder strength equal to zero. It is given by

$$S = \int \phi^*(\Delta + E)\phi + \psi^*(\Delta + E)\psi \tag{C2}$$

This field theory is free and easily solvable. The total energy momentum tensor is a sum of the energy momentum tensors of the bosonic and fermionic sectors of the theory,

$$T_{\mu\nu} = T_{\mu\nu}^b + T_{\mu\nu}^f \tag{C3}$$

As we discussed before, all the correlations of the energy momentum tensor with itself will be equal to zero because of the cancellation between fermionic and bosonic correlators. For example,

$$\langle T_{\mu\nu}(x)T_{\mu'\nu'}(0) \rangle = \langle T_{\mu\nu}^b(x)T_{\mu'\nu'}^b(0) \rangle + \langle T_{\mu\nu}^f(x)T_{\mu'\nu'}^f(0) \rangle = 0 \tag{C4}$$

On the other hand, let us define the tensor $t_{\mu\nu}$ to be

$$t_{\mu\nu} = T_{\mu\nu}^b - T_{\mu\nu}^f \tag{C5}$$

It is not difficult to check that it coincides with the tensor t as defined in appendix 1 by computing its transformation properties under $U(1|1)$. Its correlator with T is equal, on the other hand, to

$$\langle T_{\mu\nu}(x)t_{\mu'\nu'}(0) \rangle = \langle T_{\mu\nu}^b(x)T_{\mu'\nu'}^b(0) \rangle - \langle T_{\mu\nu}^f(x)T_{\mu'\nu'}^f(0) \rangle = 2 \langle T_{\mu\nu}^b(x)T_{\mu'\nu'}^b(0) \rangle \tag{C6}$$

Thus it reproduces the correlator of the purely bosonic (unitary) theory!

APPENDIX D:

Let us determine b for $U(1|1)$ current algebra. It was studied in [5] where it was shown that it describes the two dimensional Dirac fermions in the presence of random gauge potential. This algebra depends on two parameters, central extensions k and k_j . For random Dirac fermions $k = 1$ while k_j could be arbitrary positive real number related to disorder strength. More generally, k is allowed to be 1 over an arbitrary integer number. However, the physical meaning of that algebra with $k < 1$ is not completely understood in the literature.

The energy momentum tensor of such a theory is quadratic in the $U(1|1)$ currents J , j , η , and $\bar{\eta}$, and is given by

$$T = \frac{1}{2k} (Jj + \eta\bar{\eta} - \bar{\eta}\eta) + \frac{4 - k_j}{8k^2} JJ \quad (D1)$$

Of course, the central charge of this conformal field theory is equal to 0, as could be checked by computing $\langle T(z)T(0) \rangle$ [5]. The parameter b could, on the other hand, be computed in the following way.

First we find the other components of the supermultiplet T , t , ξ , and $\bar{\xi}$, by constructing expressions quadratic in currents which obey the right commutation relations,

$$\begin{aligned} \xi &= \frac{1}{4k} (\eta j + j\eta) + \frac{4 - k_j}{8k^2} \eta J \\ T &= \{\bar{\eta}, \xi\} \end{aligned} \quad (D2)$$

and

$$\begin{aligned} t &= \frac{1}{4k} jj + \frac{4 - k_j}{16k^2} (Jj + \bar{\eta}\eta - \eta\bar{\eta}) \\ 2\xi &= [\eta, t] \end{aligned} \quad (D3)$$

Note that t could only be found up to addition of other primary scalar fields of dimension 2, such as T or JJ . This ambiguity does not affect b , however.

By computing the correlation function $\langle T(z)t(0) \rangle$ with the help of the operator product expansion of the $U(1|1)$ currents taken from [5] we can determine b which turns out to be

$$b = \frac{1}{k} \quad (D4)$$

So indeed b has all the properties we might have intuitively expected. When the disorder strength $k_j = 0$, b coincides with the central charge of nonrandom Dirac fermions, $c = 1$. As the disorder strength increases b does not change. This is in agreement with the theorem proved in this paper, because the theory is critical for all values of k_j and H has to vanish everywhere. Thus b cannot change according to (6).

For other critical points $b = m$ where $k = 1/m$, m being an arbitrary integer.

We note that t is automatically holomorphic in this construction. Therefore, l_n and L_n form the algebra (9).

APPENDIX E:

We want to show that

$$\langle t_{\mu\nu} \rangle = 2 \langle T_{\mu\nu}^{\text{phys}} \rangle \quad (E1)$$

First of all, let us define what we mean by the physical free energy. The physical partition function for the disordered system with random Hamiltonian H is given by

$$Z = \int \mathcal{D}\phi^* \mathcal{D}\phi e^{-S^b} \quad (E2)$$

where S^b is a bosonic part of (A1), $S^b = \int \phi^* (H - E) \phi$ while the free energy is given by the disorder averaged logarithm of Z ,

$$F_{\text{phys}} = \langle \log Z \rangle_{\text{disorder}} \quad (E3)$$

Suppose now that the theory is defined on curved manifold with a metric $g_{\mu\nu}$. Then the expectation value of the physical energy momentum tensor can be found by differentiating F_{phys} with respect to $g_{\mu\nu}$,

$$-\frac{\delta F_{\text{phys}}}{\delta g_{\mu\nu}} = \left\langle \frac{\frac{\delta S}{\delta g_{\mu\nu}}}{Z} \right\rangle_{\text{disorder}} \equiv \langle T_{\mu\nu}^{\text{phys}} \rangle_{\text{disorder}} \quad (\text{E4})$$

Now the $1/Z$ term can be reexpressed as a fermionic path integral,

$$\frac{1}{Z} = \int \mathcal{D}\psi^* \mathcal{D}\psi e^{-S^f} \quad (\text{E5})$$

with S^f being the fermionic part of the action (A1).

Therefore, we arrive at the following formula,

$$-\frac{\delta F_{\text{phys}}}{\delta g_{\mu\nu}} = \left\langle \int \mathcal{D}[\psi, \psi^*, \phi, \phi^*] \frac{\delta S^b}{\delta g_{\mu\nu}} e^{-S} \right\rangle_{\text{disorder}} \quad (\text{E6})$$

with $S = S^b + S^f$.

Now it is not hard to see that

$$\langle t_{\mu\nu} \rangle = \langle t_{\mu\nu} + T_{\mu\nu} \rangle = 2 \left\langle \int \mathcal{D}[\psi, \psi^*, \phi, \phi^*] \frac{\delta S^b}{\delta g_{\mu\nu}} e^{-S} \right\rangle_{\text{disorder}} \quad (\text{E7})$$

We have used that $\langle T_{\mu\nu} \rangle = 0$ and the explicit construction of appendix A. (E1) obviously follows.

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